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Robust Control of Jump Parameter Systems Governed by Uncertain Chains

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Abstract— **We consider an infinite-horizon minimax optimal control problem for nonlinear stochastic uncertain systems governed by a discrete-state continuous-time chain. The chain and system dynamics are subject to uncertain disturbances. Using the large deviations theory we construct a robust stabilizing suboptimal guaranteed-performance controller. Conditions are presented under which this controller is optimal. We then present a numeric algorithm for calculating a robust (sub)optimal controller using a Markov chain approximation technique.**

I. INTRODUCTION

Minimax optimal control and robust control of uncertain stochastic systems, in which perturbations are restricted to satisfy a constraint on probability laws associated with disturbances, have been actively developed in the past decade [1], [2], [3], [4]. The majority of the results in this area deal with robust control of stochastic systems in which perturbations affect the process and/or measurement noise. This theory has been less successful in capturing some other types of uncertain perturbations occurring in stochastic systems. In this paper, we consider one such class of systems, namely, nonlinear hybrid jump parameter systems governed by a discrete state uncertain chain. In addition, dynamics of each mode of the system are uncertain.

The major novelty of this paper is the "hybrid" uncertainty model which combines the uncertainties in the discrete-event and continuous-state components of the system into a unified uncertain system model. Thus, uncertain jump parameter systems under consideration in this paper are substantially more general than those considered in the literature, e.g., see [5], [6], [7] and the references therein.

In this paper we exploit the duality between dynamic games and risk-sensitive control problems [8], [9]. This approach has proven to be useful in solving a number of stochastic LQ robust control and filtering problems [1], [3], [10], because the related risk-sensitive control problems admitted a tractable dynamic programming solution. In the general nonlinear case, such as that considered in this paper, the corresponding DP equations are not so easy to solve; e.g., see [11]. We therefore follow a more direct path and solve the risk-sensitive control problem numerically, using the techniques developed in [12].

Using the Markov chain approximation technique, the original continuous time dynamics are approximated by "locally consistent" Markov chain dynamics in discrete-time [12]. Under these local consistency conditions, certain weak convergence results can be established that allow the original non-linear risk-sensitive control problem to be approximated by an analogous risk-sensitive control problem on these approximating Markov chain dynamics. This allows us to develop numerically tractable algorithms for calculating our desired controller.

II. MATHEMATICAL PRELIMINARIES

A. Uncertain Markov chains

We first give a review of uncertain Markov chains [9]. Let Ω^r be a Skorokhod space of cadlag functions $[0, +\infty) \to \mathbb{E} = \{1, \ldots, N\}$ [13] endowed with the Borel σ-algebra $\mathcal{F}^r$. Let P^r and $\mathbf{r}(\cdot)$ be a probability measure on $(\Omega^r, \mathcal{F}^r)$ and a Markov chain with values in $\mathbb{E}$, respectively. Let us define the natural filtration $\{\mathcal{F}_t^r, t \geq 0\}$ generated by mappings M_t, $t \geq 0$ of the form $M_t[\mathbf{r}(\cdot)] = r(t)$, for all $r(.) \in \Omega^r$. The probability space $(P^r, \Omega^r, \mathcal{F}^r)$ is assumed to be complete.

The chain $\mathbf{r}(\cdot)$ is assumed to be homogeneous, stationary, irreducible and aperiodic [14]. In particular, this implies that its state transition probabilities satisfy [15]

$$P^r(\mathbf{r}(t+\Delta t) = j|\mathbf{r}(t) = i) = \begin{cases} \pi_{ji}\Delta t + o(\Delta t), & j \neq i, \\ 1 + \pi_{ii}\Delta t + o(\Delta t); & \end{cases} \tag{1}$$

here $\pi_{ij} \geq 0$, and $\pi_{ii} = -\sum_{j \neq i} \pi_{ji}$. Let $\Pi := [\pi_{ij}]$. For each $j \in \mathbb{E}$, the probability rate of $\mathbf{r}$ jumping to the state j at time t defines the mapping $b^j : [0, \infty] \times \Omega^r \to \mathbf{R}^+$,

$$b_t^j(r) = \pi_{ji} \text{ if } r(t) = i \neq j \quad \text{and} \quad b_t^j(r) = 0 \text{ otherwise.} \tag{2}$$

Also, define a counting process $N_t^j(r)$ in $(\Omega^r, \mathcal{F}^r, P^r)$, which is known to admit the decomposition [9], [16] $N_t^j = \int_0^t b_s^j ds + v_t^j$; v_t^j is a martingale.

Following [9], consider perturbations of the measure P^r characterized in terms of $\mathcal{F}_t^r$-progressively measurable processes δ_t^j, $j = 1, \ldots, N$, satisfying the conditions:

($\delta 1$) For all $T > 0$, there exists a probability measure $Q^{r,T}$ on $(\Omega^r, \mathcal{F}_T^r)$ under which $\tilde{v}_t^j = N_t^j - \int_0^t \delta_s^j b_s^j ds$ is a local martingale with respect to the filtration $\{\mathcal{F}_t^r, t \in [0, T]\}$.

($\delta 2$) For each $j \in \mathbb{E}$, $\int_0^T b_s^j \left(1 - \sqrt{\delta_s^j}\right)^2 ds < \infty$ $Q^{r,T}$-a.s..

($\delta 3$) Conditions ($\delta 1$) and ($\delta 2$) ensure that the probability measures $Q^{r,T}$ are absolutely continuous with respect to the probability measure $P^{r,T}$, $Q^{r,T} \ll P^{r,T}$ [9], [16]; the latter is the restriction of the measure P^r to $\mathcal{F}_T^r$. In addition to properties ($\delta 1$) and ($\delta 2$), we restrict attention to those perturbation processes δ_t for which

$$h(Q^{r,T} \| P^{r,T}) := \mathbf{E}^{Q^{r,T}} \log \frac{dQ^{r,T}}{dP^{r,T}} < \infty; \tag{3}$$

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here $h(Q\|P)$ denotes the relative entropy between probability measures Q and P [8],

$$h(Q\|P) := \begin{cases} \mathbf{E}^Q \log\left(\frac{dQ}{dP}\right) & Q \ll P,\, \log\left(\frac{dQ}{dP}\right) \in L_1(dQ) \\ +\infty & \text{otherwise,} \end{cases}$$

and $\mathbf{E}^Q$ denotes the expectation with respect to Q.

Conversely, for any probability measure $Q^{r,T}$ on $(\Omega^r, \mathcal{F}^r_T)$ such that $h(Q^{r,T}\|P^{r,T}) < \infty$, an $\mathcal{F}^r_t$-progressively measurable process δ^j_t can be found which satisfies the above conditions [9]. This allows us to proceed regarding processes $[\delta^j_{(\cdot)}]_{j=1}^N$, or equivalently, collections $\{Q^{r,T}, T \geq 0\}$, satisfying conditions ($\delta 1$), ($\delta 2$), and ($\delta 3$) as uncertain perturbations of the chain $\mathbf{r}(t)$.

We note that properties ($\delta 1$), ($\delta 2$) and ($\delta 3$) describe a rich class of perturbations of the reference Markov chain which include nonhomogeneous, or nonstationary processes.

B. Jump parameter systems governed by uncertain chains

1) Nominal system governed by a Markov chain: We will consider a class of stochastic systems driven by the Markov chain $\mathbf{r}$ and a Brownian motion $\mathbf{w}$, both processes being defined on a joint probability space. Such a probability space can be thought of as the product-space $(\Omega, \mathcal{F}, P)$, where $\Omega = \Omega^r \times \Omega^w$, $P = P^r \times P^w$ and $\mathcal{F}$ is the completion of $\mathcal{F}^r \times \mathcal{F}^w$. Here, $(\Omega^r, \mathcal{F}^r, P^r)$ is the canonical probability space of the Markov chain $\mathbf{r}$ constructed in Section II-A, and $(\Omega^w, \mathcal{F}^w, P^w)$ denotes the canonical noise space of a Wiener process $\mathbf{w}(t)$.

On $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P)$, consider a stochastic system

$$\begin{aligned} dx(t) &= f(x(t), u(t), \mathbf{r}(t))dt + \sigma(x(t), \mathbf{r}(t))d\mathbf{w}(t), \quad (4) \\ z(t) &= g(x(t), u(t), \mathbf{r}(t)), \qquad x(0) = x_0 \in \mathbf{R}^n, \end{aligned}$$

$\mathbf{r}(t)$ and $\mathbf{w}(t)$ are independent. Here, $x(t) \in \mathbf{R}^n$ is the state, $z(t) \in \mathbf{R}^q$ is the uncertainty output, $u(t)$ is the control input which takes values in a compact metric space U, and f, g, σ are continuous and globally Lipschitz in x, uniformly in u mappings $\mathbf{R}^n \times U \times \mathbb{E} \to \mathbf{R}^n$, $\mathbf{R}^n \times U \times \mathbb{E} \to \mathbf{R}^q$, $\mathbf{R}^n \times \mathbb{E} \to \mathbf{R}^{n\times p}$. Also, $\sigma(x,e)\sigma'(x,e) \geq \sigma_0 I > 0$ for all $x \in \mathbf{R}^n$, $e \in \mathbb{E}$. Furthermore, for each $e \in \mathbb{E}$, $\rho > 0$, the set $\{x : \sup_{u\in U} g(x,u,e) \leq \rho\}$ is compact.

Following [11], we focus on the set $\mathcal{U}_d$ of deterministic (nonrandomized) Markov controls of the form $u(t) = \mathcal{K}(x(t), \mathbf{r}(t))$, $\mathcal{K}$ is a measurable function $\mathbf{R}^n \times \mathbb{E} \to U$.

2) Perturbed jump parameter systems: As in [1], [2], [3], the probability measure P is not fixed, rather collections of probability measures $\{Q^T, T > 0\}$ on $(\Omega, \mathcal{F}_T)$ will be considered such that $h(Q^T\|P^T) < \infty$; P^T is the restriction of P to $(\Omega, \mathcal{F}_T)$. For each $T > 0$, the set of such probability measures Q^T will be denoted $\mathcal{P}_T$. The system (4) under $Q^T \in \mathcal{P}_T$ will be regarded as a perturbed system. Accordingly, the system (4) under P^T is the nominal system.

Although admissible perturbations of the nominal model remain unknown, they are usually assumed to be bounded in magnitude in some sense. Following [1], [2], [3], we use the relative entropy functional to measure the size of admissible perturbations.

Definition 1 *Let $d > 0$ be a given constant. A collection of probability measures $\{Q^T, T \geq 0\}$ is said to define an admissible perturbation of the system (4) if for each $T > 0$, $Q^T \in \mathcal{P}_T$ and there exists a nonnegative constant $\varepsilon_T = o(1/T)$ such that the measure Q^T satisfies the constraint*

$$\sup_{T'>T} \frac{1}{T'} \left(h(Q^{T'}\|P^{T'}) - \mathbf{E}^{Q^{T'}} \int_0^{T'} \|z\|^2 dt \right) \leq d + \varepsilon_T. \quad (5)$$

III. Robust control problem

We study a robust control problem for nonlinear systems (4) whose dynamics evolve in uncertainty probability spaces $(\Omega, \mathcal{F}_T, Q^T)$, $T > 0$, under the uncertainty constraint (5). To evaluate the system performance we will use the cost

$$J(u, Q) = \varlimsup_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{E}^{Q^T} c(x(t), u(t), \mathbf{r}(t))dt; \quad (6)$$

Here $\varlimsup$ denotes $\limsup$. The running cost $c(x,u,e) \geq 0$ is continuous in (x,u) for each $e \in \mathbb{E}$, and the set $\{x : \sup_{e\in\mathbb{E}, u\in U} c(x,u,r) \leq \rho\}$ is compact for each $\rho > 0$. The variable Q refers to an admissible collection of measures $\{Q^T, T > 0\}$; the set of such perturbations is denoted Ξ_d. Therefore, the objective is to find a suboptimal solution u^* to the optimization problem

$$\inf_u \sup_{Q\in\Xi_d} J(u, Q) \leq \sup_{Q\in\Xi_d} J(u^*, Q). \quad (7)$$

The suboptimal controller is sought in the class of suitable deterministic Markov control policies providing a certain robust closed loop stability. Also, the controller u^* is desirable which gives a tight bound on the worst-case performance so that inequality (7) becomes the exact equality.

In [11], solutions to a related risk-sensitive control problem (see (9) below) for the nominal system were sought in the class of deterministic Markov controls $u \in \mathcal{U}_d$ for which the closed-loop process $(x(t), \mathbf{r}(t))$ (4) has a unique invariant probability measure on $\mathbf{R}^n \times \mathbb{E}$; such control strategies were termed in [11] stabilizing policies; see Definition 3 below. A similar blanket assumption of positive recurrence was used in [17]. However, in the presence of uncertain perturbations considered in this paper an invariant measure may not exist. To account for this fact, we present the stability property relevant to the uncertain system (4).

Definition 2 *The closed loop system corresponding to a control policy $u(\cdot) \in \mathcal{U}_d$ is absolutely stable if there exists a $\beta > 0$ such that for all admissible perturbations,*

$$\varlimsup_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{E}^{Q^T} \|x(t)\|^\beta dt \leq \gamma, \quad (8)$$

where the constant $\gamma > 0$ is independent of $Q^T \in \Xi_d$.

The following risk-sensitive control problem will be instrumental in the derivation of a solution to the problem (7),

$$\inf_{u\in\mathcal{U}_d} \mathcal{J}_\theta(u), \quad (9)$$

$$\mathcal{J}_\theta(u) := \varlimsup_{T\to\infty} \frac{\theta}{T} \log \mathbf{E} \exp\left[\frac{1}{\theta}\int_0^T c_\theta(x(t), u(t), \mathbf{r}(t))dt\right],$$

$$c_\theta(x,u,e) := c(x,u,e) + \theta\|g(x,u,e)\|^2.$$

Theorem 1 *Suppose there exists an admissible controller u_θ such that $V_\theta := \inf_{u\in\mathcal{U}_d} \mathcal{J}_\theta(u) < +\infty$. Then this controller solves the guaranteed cost control problem (4), (5), (7), and $\sup_{Q\in\Xi_d} J(u_\theta, Q) \le V_\theta + \theta d$. Furthermore,*

$$\inf_{u\in\mathcal{U}_d} \sup_{Q\in\Xi_d} J(u,Q) \le \inf_\theta (V_\theta + \theta d); \qquad (10)$$

the infimum on the RHS is taken over the set of parameters $\theta \ge 0$ for which the problem (9) admits a solution. If $c(x,u,i) \ge \alpha \|x\|^\beta$ for all $i \in \mathbb{E}$, then the controller obtained from (10), is an absolutely stabilizing controller.

We will now show that the right-hand side of (10) gives the optimal worst-case performance, provided robust controllers of interest are those which exercise *stationary stabilizing* Markov state-feedback.

Definition 3 *A control $u \in \mathcal{U}_d$ is said to be stationary stabilizing for the nominal system (4) if there exists a unique invariant probability measure μ^u on $\mathbf{R}^n \times \mathbb{E}$, i.e.,*

$$\sum_{i\in\mathbb{E}} \int_{\mathbf{R}^n} P\left(\begin{matrix} x(t) \in B, \\ \mathbf{r}(t) = j \end{matrix} \middle| \begin{matrix} x(0) = x, \\ \mathbf{r}(0) = i \end{matrix} \right) \mu^u(dx, i) = \mu^u(B \times j)$$

for any Borel set $B \in \mathbf{R}^n$. The set of all stationary stabilizing controls will be denoted $\mathcal{U}_{ds}$.

To present a rigorous formulation of the necessity result to complement that of Theorem 1, some additional technical conditions on control solutions are needed. These conditions are taken from [11].

Condition 1 *Let $P^u(t, (x,i), B \times \{j\})$ denote the transition probability function for the nominal composite Markov process $(x(t), \mathbf{r}(t))$. Let a control solving Theorem 1 be stationary stabilizing, $u = u(x,i) \in \mathcal{U}_{ds}$, and let there exist a $\tau^u > 0$, a σ-finite measure η^u on $\mathbf{R}^n \times \mathbb{E}$ and a function $q^u(x,i,y,j)$ such that*

(a) $q^u(x,i,y,j) > 0$ for η^u-almost all $(x,i) \in \mathbf{R}^n \times \mathbb{E}$;
(b) $P^u(\tau^u, (x,i), B \times \{j\}) = \int_B q^u(x,i,y,j)\eta^u(dy,j)$;
(c) For all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x - x'| < \delta$ then $\sum_{j=1}^N \int_{\mathbf{R}^n} |q^u(x,i,y,j) - q^u(x',i,y,j)|\eta^u(dy,j) < \varepsilon$.

Consider a set $\mathcal{V}_d$ of two-component nonrandomized Markov *disturbances* $v = [v_1\, v_2] : \mathbf{R}^n \times \mathbb{E} \to \mathbf{R}^{p+1}$, whose second component v_2 is a measurable positive function $\mathbf{R}^n \times \mathbb{E} \to \mathbf{R}^1$. Also define $\tilde{\mathbf{r}}$ to be a continuous time Markov chain taking values in $\mathbb{E}$, with generator matrix entries

$$\tilde\pi_{ji} = \tilde\pi_{ji}(x, v_2) = \pi_{ji} \frac{v_2(x,j)}{v_2(x,i)}, \quad j \ne i, \quad \text{and}$$
$$\tilde\pi_{ii} = \tilde\pi_{ii}(x, v_2) = -\sum\nolimits_{l\ne i} \tilde\pi_{li}(x, v_2). \qquad (11)$$

To formulate the second assumption of [11], consider the system in $(\Omega, \mathcal{F}, P)$

$$\begin{aligned} d\tilde x(t) &= (f(\tilde x(t), \tilde u(t), \tilde{\mathbf{r}}(t)) + \sigma(\tilde x(t), \tilde{\mathbf{r}}(t))\tilde v_1(t))dt \\ &\qquad + \sigma(\tilde x(t), \tilde{\mathbf{r}}(t))d\mathbf{w}(t), \quad (12) \\ \tilde z(t) &= g(\tilde x(t), \tilde u(t), \tilde{\mathbf{r}}(t)), \qquad \tilde x(0) = x_0 \in \mathbf{R}^n, \\ \tilde u(t) &:= u(\tilde x(t), \tilde{\mathbf{r}}(t)), \quad \tilde v_1(t) := v_1(\tilde x(t), \tilde{\mathbf{r}}(t)). \end{aligned}$$

Condition 2 *For a stationary stabilizing control solving Theorem 1, $u \in \mathcal{U}_{ds}$, there exists a nonnegative function $\phi^u \in C^2(\mathbf{R}^n \times \mathbb{E})$ such that*

(i) $\lim_{|x|\to\infty} \phi^u(x,i) = \infty$;
(ii) There exists $\rho > 0$, $\varepsilon > 0$ such that $(A^{u,v}\phi^u)(x,i) < -\varepsilon$ for $|x| > \rho$ and $i \in \mathbb{E}$, and $\left|\frac{\partial \phi^u}{\partial x}(x,i)\right|^2 > \sigma_0^{-1}$; $A^{u,v}$ is the infinitesimal generator of (12).
(iii) $\phi^u(x,i)$ and $\left|\frac{\partial \phi^u}{\partial x}(x,i)\right|^2$ have polynomial growth in x.

Conditions 1 and 2 guarantee that for any Markov disturbance $v(\cdot) \in \mathcal{V}_d$, the system (12) has a unique invariant measure $\mu^{u,v}$ [11]. Further, define

$$H(x,i,u,v_2) = \sum_{j\in\mathbb{E}} \left[\tilde\pi_{ji}(x, v_2) \log v_2(x,j) - \pi_{ji} \frac{v_2(x,j)}{v_2(x,i)} \right],$$
$$C_\theta(x,i,u,v) = c_\theta(x,i,u) - \frac{1}{2}\|v_1\|^2 - H(x,i,u,v_2),$$

and consider the functional associated with the system (12),

$$\bar J(u,v) = \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T C_\theta(\tilde x(t), \tilde{\mathbf{r}}(t), \tilde u(t), \tilde v(t))dt.$$

In view of Condition 2, we have

$$\bar J(u,v) = \sum\nolimits_{i=1}^N \int_{\mathbf{R}^n} C_\theta(x,i,u,v)\mu^{u,v}(dx,i).$$

We now denote the elements of $\mathcal{U}_{ds}$ that also meet the conditions 1 and 2 as $\bar{\mathcal{U}}_{ds}$. Theorem 4 of [11] claims that

$$\mathcal{J}_\theta(u) = \sup\nolimits_{v\in\mathcal{V}_d} \bar J(u,v) \qquad \forall u \in \bar{\mathcal{U}}_{ds}. \qquad (13)$$

We are in a position to present a result which can be regarded as a converse result to Theorem 1. Let $V_\theta^0 := \inf_{u\in\bar{\mathcal{U}}_{ds}} \mathcal{J}_\theta(u)$.

Theorem 2 *(i) If the set $\{\theta : V_\theta^0 < \infty\}$ is not empty then the robust minimax control problem given on the left-hand side of (7) admits a solution in the class of deterministic stationary stabilizing controls $\bar{\mathcal{U}}_{ds}$.*

(ii) The robust minimax control problem given on the left-hand side of (7) admits a solution in the class $\bar{\mathcal{U}}_{ds}$, only if the set $\{\theta : V_\theta^0 < \infty\}$ is not empty. In this case,

$$\inf_{u\in\bar{\mathcal{U}}_{ds}} \sup_{Q\in\Xi_d} J(u,Q) = \inf_{\theta: V_\theta^0 < \infty} (V_\theta^0 + \theta d). \qquad (14)$$

IV. NUMERIC SOLUTIONS USING A MARKOV CHAIN APPROACH

The dynamic programming equation for the risk-sensitive control problem (9) is given in [11]. Finding its direct solution is difficult in general. This motivates consideration of a numeric approximation approach based on the Markov chain scheme of [12]. In this approach, the risk-sensitive control problem for unperturbed continuous-time hybrid dynamics (4), (1) is approximated by a risk-sensitive control problem on controlled discrete-time Markov chain dynamics. To simplify presentation, we have only considered approximations of dynamics with diagonal $\sigma(x,e)$. Approximations for dynamics with non-diagonal $\sigma(x,e)$ can be developed in a similar way [12, pp. 108–110].

In the first two parts of this section we will need to assume, as a technical condition, that f and σ are bounded for all $x \in R^n$, $u \in U$ and $e \in \mathbb{E}$. In practice, this assumption is easily satisfied due to a practical requirement to approximate the dynamics on a bounded region of state-space (on a bounded region, f, σ are bounded under the original assumptions on the dynamics).

A. Construction of a Markov chain approximation

We follow the approach taken in [12, Ch. 5]. Let us define $d_i \in R^n$ to be indicator column vectors, $d_{ii} = 1$, $d_{ij} = 0$ for $j \neq i$. Then a uniform grid of size h is defined as follows: $\bar{S}^h = \{x : x = h\sum_i d_i m_i : m_i = 0, \pm 1, \pm 2, \ldots\}$.

To approximate our original unperturbed dynamics (1), (4) we consider two discrete-time Markov chains, $\bar{x}_k \in \bar{S}^h$, $\bar{r}_k \in \mathbb{E}$ for $k = 0, 1, \ldots$, defined on complete probability spaces $(\bar{\Omega}^{hx}, \bar{\mathcal{F}}^{hx}, \bar{P}^{hx})$, $(\bar{\Omega}^{hr}, \bar{\mathcal{F}}^{hr}, \bar{P}^{hr})$, respectively. These probability spaces will be interpreted as approximations for $(\Omega^r, \mathcal{F}^r, P^r)$, $(\Omega^w, \mathcal{F}^w, P^w)$. We assume that these spaces are endowed with the natural filtrations $\bar{\mathcal{F}}^x_k$, $\bar{\mathcal{F}}^r_k$, $k \geq 0$, generated by the processes $\bar{x}_k$, $\bar{r}_k$, respectively.

Below we will define appropriate transition probabilities for these approximating Markov chains. $\bar{P}^{hx}(\bar{y}|\bar{x}, u, e)$ will denote the transition probability from state $\bar{x}_k = \bar{x}$ to state $\bar{x}_{k+1} = \bar{y}$ when $\bar{r}_k = e$ and control action $u \in U$ is applied. $\bar{P}^{hr}(j|e)$ will denote the transition probability from $\bar{r}_k = e$ to $\bar{r}_{k+1} = j$. From these two chains we will define a composite chain $(\bar{x}_k, \bar{r}_k)$ on the product probability space $(\bar{\Omega}^h, \bar{\mathcal{F}}^h, \bar{P}^h)$ where $\bar{\Omega}^h = \bar{\Omega}^{hx} \times \bar{\Omega}^{hr}$, $\bar{P}^h = \bar{P}^{hx} \times \bar{P}^{hr}$ and $\bar{\mathcal{F}}^h$ is the completion of $\bar{\mathcal{F}}^{hx} \times \bar{\mathcal{F}}^{hr}$.

To develop an approximating Markov chain consistent with our original unperturbed continuous-time dynamics we define a fixed time step $\Delta\bar{t}^h = \frac{h^2}{D^h}$, where $D^h \geq \sigma_0 > 0$ is defined by

$$D^h = \max_{\bar{x} \in \bar{S}^h} \max_{u \in U, e \in \mathbb{E}} \left\{ \sigma^2(\bar{x}, e, u) + h \sum_k |f_k(\bar{x}, e, u)| \right\}.$$

We assume h is small enough so that $0 \leq |\pi_{ee}| \Delta\bar{t}^h < 1$.

As a first step in constructing a suitable Markov chain approximation, we consider approximation of (4) for each mode value. For each $e \in \mathbb{E}$, consider a discrete-time Markov chain with probabilities of transition to neighboring states

$$\bar{P}^{hx}\left(\bar{x} \pm h d_j \middle| \bar{x}, u, e\right) = \frac{\sigma^2(\bar{x}, e, u)/2 + h f_j^{\pm}(\bar{x}, e, u)}{D^h} \tag{15}$$

where $j \in \{1, \ldots, n\}$, $f_j(\cdot)$ is the jth component of $f(\cdot)$, $f_j^+ = \max[f_j, 0]$ and $f_j^- = \max[-f_j, 0]$. Further, we assume that the probability of transition to non-neighboring states is zero. With this choice of transition probabilities, the probability of remaining in a particular state is given by

$$\bar{P}^{hx}\left(\bar{x} \middle| \bar{x}, u, e\right) = 1 - \sum\nolimits_{\bar{y} \neq \bar{x}} \bar{P}^{hx}\left(\bar{y} \middle| \bar{x}, u, e\right).$$

To complete our approximation of the original unperturbed dynamics we define a discrete-time Markov chain representing the mode process, $\bar{r}_k$, with the transition probability matrix $\bar{P}^{hr} = \exp(\Pi \Delta\bar{t}^h)$.

We define the processes $\bar{x}_k$ and $\bar{r}_k$ to have independent increments so that for the composite Markov chain $(\bar{x}, \bar{r})$ we define a product probability measure under which

$$P^h\left((\bar{y}, j) \middle| (\bar{x}, e), u\right) = \bar{P}^{hr}(j|e) \bar{P}^{hx}\left(\bar{y} \middle| \bar{x}, u, e\right).$$

$\mathbf{E}^h[\cdot]$ will denote the expectation with respect to $P^h(\cdot)$ (and similarly define conditional expectations).

With the composite Markov chain $(\bar{x}, \bar{r})$ we associate a continuous parameter interpolation process

$$X^h(t) = \bar{x}_k \text{ and } R^h(t) = \bar{r}_k, \quad t \in [k\Delta\bar{t}^h, (k+1)\Delta\bar{t}^h)$$

It is this continuous parameter interpolation process that approximates our original unperturbed continuous-time hybrid dynamics (when $\sigma(x, e)$ is diagonal for all $x \in R^n$, $e \in \mathbb{E}$) in the sense given in the next lemma.

Lemma 1 *Consider the unperturbed dynamics (4), (1) and additionally assume that f, σ are bounded, and that $\sigma(x, e)$ is diagonal for all x, e. Let $\bar{x}^+ = X^h((k+1)\Delta\bar{t}^h)$, $\bar{x} = X^h(k\Delta\bar{t}^h)$, $\bar{r}^+ = R^h((k+1)\Delta\bar{t}^h)$ and $\bar{r} = R^h(k\Delta\bar{t}^h)$. The continuous parameter interpolation process $(X^h(t), R^h(t))$ with transition probabilities $\bar{P}^h\left((\bar{y}, j) \middle| (\bar{x}, e), u\right)$ is locally consistent with the unperturbed dynamics in the sense that*

$$\mathbf{E}^h\left[\bar{x}^+ - \bar{x} | \bar{r}, \bar{x}, u\right] = f(\bar{x}, \bar{r}, u)\Delta\bar{t}^h + o(\Delta\bar{t}^h),$$

$$\mathbf{E}^h\left[(\bar{x}^+ - \bar{x})(\bar{x}^+ - \bar{x})' | \bar{r}, \bar{x}, u\right] = \sigma(\bar{x}, \bar{r})\sigma(\bar{x}, \bar{r})'\Delta\bar{t}^h + o(\Delta\bar{t}^h),$$

$$\bar{P}^{hr}\left(R^h(k\Delta\bar{t}^h + \Delta\bar{t}^h) = j | R^h(k\Delta\bar{t}^h) = e\right) = \left[e^{\Pi\Delta\bar{t}^h}\right]^{je},$$

$$P^h\left(\bar{x}^+, \bar{r}^+ | \bar{r}, \bar{x}, u\right) = P^h\left(\bar{x}^+ | \bar{r}, \bar{x}, u\right) P^h\left(\bar{r}^+ | \bar{r}, \bar{x}, u\right),$$

$$\sup_t \mathbf{E}^h[|X^h(t + \Delta\bar{t}^h) - X^h(t)|] \xrightarrow{h} 0,$$

$$\sup_t \mathbf{E}^h[|R^h(t + \Delta\bar{t}^h) - R^h(t)|] \xrightarrow{h} 0.$$

Our choice of approximating dynamics is not the only choice that meets these local consistency conditions. For example, locally consistent dynamics in which $\Delta\bar{t}^h$ varies over the state space and with control variable choice are also possible. However, these types of approximations are avoided because they are not well suited to time-averaged control problems [12]. Moreover, our assumption that σ is diagonal can be relaxed by using other Markov chain approximation structures.

The established local consistency results allow us to apply most of the results from [12, Ch. 11] in a fair routine manner. Before proceeding to establish the main result of this section we need to introduce the concept of a relaxed control. Consider the σ-algebras $\mathcal{B}(U)$ and $\mathcal{B}(U \times [0, \infty))$ defined as the collections of Borel subsets of U and $U \times [0, \infty)$, respectively. An admissible relaxed control is then a random variable $m(.)$ taking values in the space of Borel measures on $\mathcal{B}(U \times [0, \infty))$ and such that $m(U \times [0, t)) = t$ for all $t \geq 0$ and $m(A \times [0, t])$ is $\mathcal{F}_t$-adapted for all $A \in \mathcal{B}(U)$. We can then define a 'derivative' $m_t(.)$, where $m_t(A)$ is $\mathcal{F}_t$-adapted for all $A \in \mathcal{B}(U)$, such that

$$m(B) = \int_{U \times [0, \infty)} I_{(\alpha, t) \in B} m_t(d\alpha) dt \quad \text{w.p. } 1$$

for all $B \in \mathcal{B}(U \times [0,\infty))$ and such that for each t, $m_t(.)$ is a random measure on $\mathcal{B}(U)$ satisfying $m_t(U) = 1$ with probability 1. The space of such relaxed controls can then be metrized using the Prohorov metric [12].

B. Weak convergence of Markov chain approximations

To connect the approximation Markov chain dynamics with the original unperturbed dynamics we establish weak convergence of the approximation as $h \to 0$.

Within the next theorem, motivated by [12], we will assume that for each $h > 0$, there is a probability space on which are defined a filtration $\mathcal{F}_t^h$, a process $w^h(.)$, an admissible relaxed control $m^h(.)$ and solution processes $(X^h(.), R^h(.))$. The $w^h(.)$ and $R^h(.)$ are independent and adapted to $\mathcal{F}_t^h$ where the filtration satisfies $\mathcal{F}_t^h \supset \mathcal{F}(X^h(s), R^h(s), m_s^h(.), w^h(s), s \le t)$. Thus, $X^h(.)$ satisfies

$$X^h(t) = X^h(0) + \int_0^t \int_U f(X^h(s), R^h(s), \alpha) m_s^h(d\alpha) ds + \int_0^t \sigma(X^h(s), R^h(s)) dw^h(s) + o(h).$$

This representation follows from Lemma 1 and noting that for $h > 0$ these integrations can be expressed as summations.

Theorem 3 *Consider dynamics (4), (1) and assume that f, σ are bounded. Suppose the local consistency conditions of Lemma 1 hold. Let $X^h(0) \xrightarrow{h} x_0$. Then any sequence $\left\{X^h(.), R^h(.), m^h(.), w^h(.)\right\}$ as $h \to 0$ is tight. Let $(X(.), R(.), m(.), w(.))$ denote the limit of a weakly convergence subsequence. Define $\mathcal{F}_t = \mathcal{F}(X(s), R(s), m_s(.), w(s), s \le t)$, $\mathcal{F}_t^r = \mathcal{F}(R(s), s \le t)$. Then $w(.)$ and $R(.)$ are mutually independent standard $\mathcal{F}_t$-Wiener process and $\mathcal{F}_t^r$-Markov chain, respectively, $m_.(.)$ is admissible with respect to $(w(.), R(.))$, $X(0) = x_o$, $x(.) = X(.)$ satisfies the dynamics (4) weakly and $R(.)$ satisfies the dynamics (1) weakly.*

We next outline the solution to the appropriate discrete-time risk-sensitive Markov chain problem.

C. Risk-sensitive control of discrete-time Markov chains

Consider a discrete-time, stationary, controlled Markov chain $\bar{z}_k$ that takes values from a finite state space $\bar{\mathcal{Z}}$, that is $\bar{z}_k \in \bar{\mathcal{Z}}$ for all $k = 0, 1, \ldots$. Further, we consider controls from a compact set $\bar{U}$, a set of Markov (possibly randomized) policies $\bar{\mathcal{U}}_R = \{u(.) : \bar{\mathcal{Z}} \to P(\bar{U})\}$, and a set of stationary deterministic Markov policies $\bar{\mathcal{U}}_D = \{u(.) : \bar{\mathcal{Z}} \to \bar{U}\}$.

For each $u \in \bar{U}$ we have controllable transition probability matrices $\bar{P}^z(i|j,u)$ that are assumed irreducible and aperiodic for each control. Further we assume that $\bar{P}^z(i|j,u)$ is continuous in u and $\bar{P}^z(i|i,u) > 0$ for all $i \in \bar{\mathcal{Z}}$ and all $u \in \bar{U}$. Associated with these controlled dynamics we consider a non-negative bounded one-stage cost $\bar{c}_\theta(\bar{z}, u)$.

The risk-sensitive control problem for these discrete-time Markov chain dynamics is to design a control policy $u(.) \in \bar{\mathcal{U}}_R$ that minimizes the cost $\bar{\mathcal{J}}^\theta(\bar{z}_0, u)$ from initial state $\bar{z}_0$,

$$\bar{\mathcal{J}}^\theta(\bar{z}_0, u) = \overline{\lim_{m\to\infty}} \frac{\theta}{m} \log \mathbf{E}^{\bar{P}^z} e^{\frac{1}{\theta}\sum_{k=0}^m \bar{c}_\theta(\bar{z}_k, u(\bar{z}_k))}. \quad (16)$$

If $\bar{c}_\theta(.,.) = \Delta\bar{t}^h c_\theta(.,.,.)$, then $\bar{\mathcal{J}}^\theta(\bar{z}_0, u)/\Delta\bar{t}^h$ is a suitable discrete-time discrete-state approximation for the risk-sensitive control problem (9).

According to [18], suppose there is a $\lambda > 0$ and a $\bar{\psi} : \bar{\mathcal{Z}} \to R$ which is a strictly positive function such that

$$\lambda\bar{\psi}(i) = \min_{u\in\bar{U}} \left\{ e^{\frac{1}{\theta}\bar{c}_\theta(i,u)} \sum_{j\in\bar{\mathcal{Z}}} \bar{P}^z(j|i,u)\bar{\psi}(j) \right\} \quad (17)$$

for each $i \in \bar{\mathcal{Z}}$. Then these dynamic programming equations solve the discrete-time Markov chain risk-sensitive control problem in that, $\inf_{u(.)\in\bar{\mathcal{U}}_R} \bar{\mathcal{J}}^\theta(j, u(.)) = \theta \log_n \lambda$ for every $j \in \bar{\mathcal{Z}}$. If $u(i) \in \bar{U}$ is the minimizing control in (17) then $u(.) \in \bar{\mathcal{U}}_D \subset \bar{\mathcal{U}}_R$ is the optimal control policy for this problem [18, Thm 2.1]. Further, from our irreducible and aperiodic assumption on $\bar{P}^z$, we know that the minimizing control meets a natural analogy of Conditions 1 and 2.

Further, under the above assumptions, a suitable λ and $\bar{\psi}$ always exists [18, Thm 2.2] and convergence of a suitable value iteration for (17) can be established [18, Thm 2.3]. Although unaware of convergence results for a policy iteration algorithm when $\bar{U}$ is a compact set, we prefer to use a policy iteration solution technique in our numeric algorithms.

V. Example: Control of a power generation node

We consider the problem of controlling one node of an interconnected power system. This example and the main dynamic model parameters are given in [19].

We assume that no direct information about other power generation nodes is available, but that these other nodes can induce changes to power generation dynamics of the node under consideration. The influence of the power generation grid is represented through the presence of a discrete-valued process $\mathbf{r}(t) \in \mathbb{E} = \{1, 2\}$ which corresponds to possible induced changes in the dynamics of power generation of the node being controlled. These changes are due to changes in loads connected to the grid; see [19] for details.

In our controlled dynamic model of one power generation node: let $x(t) = [\Delta\delta(t), \Delta\dot{\delta}(t)]'$ be the change in generator rotor angle (relative to the reference angle) and its rate of change, respectively; let $u(t) = [\Delta P(t), \Delta Q(t)]'$ be changes in real and reactive input power (control inputs); and let $\mathbf{r}(t)$ describe the dynamics of the power grid interference due to load changes. We assume $|\Delta P(t)| \le 1$ and $|\Delta Q(t)| \le 1$; this constraint reflects physical limitations on the amount of power which can be received from mechanical drives.

Consider the following hybrid dynamic description of a power generation node:

$$dx(t) = (A_{\mathbf{r}(t)}x(t) + B_{\mathbf{r}(t)}u(t))dt + \sigma d\mathbf{w}(t). \quad (18)$$

In accordance with the approach outlined in Section II-B, we assume that the system (18) is defined in a true probability space. The latter space is associated with physical disturbances, and is therefore uncertain. In the reference probability space however, $\mathbf{w}(t)$ is a standard Wiener process with unity covariance, σ is a diagonal 2×2 matrix with both diagonal elements equal to $\sqrt{0.1}$, $\mathbf{r}(t)$ is a Markov chain process with 2×2 transition probability rate matrix (1) in which $\pi_{ij} = 0.1$ for $i \neq j$, and

$$[A_1|A_2] = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ -1.810 & -0.476 & -1.841 & -0.476 \end{array}\right]$$
$$[B_1|B_2] = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0.476 & -7.323 & 0.476 & -6.435 \end{array}\right].$$

We assume that there is some uncertainty in our model of the grid interconnection dynamics in that admissible perturbations of the class Ξ_d with $d = 0.05$ (with $g(.,.,.) = 0$) are present in our system. According to Definition 1, this constraint captures unmodeled dynamics which have bounded power, and also imprecisely known and time-varying deviations of transition-probability rates from the nominal values specified above. Consider a running cost $c(x, u, e) = |x|^2 + |u|^2$. With this choice of running cost, consider the worst-case infinite-horizon cost as defined by (6). Our robust control problem on these power generation node dynamics is to find a suboptimal control law u^* as defined by (7). Note that with the above choice of the running cost, Theorem 1 will yield the robust closed-loop stability.

As described by Theorem 1, our robust suboptimal control design is achieved through a line-search (over θ) of $V_\theta + \theta d$. Using the Markov chain approximation technique presented in Section IV (with $\bar{N}h = 1.5$ and $h = 3/28$) we numerically solved the risk-sensitive control problem for various θ to enable us to approximately find the θ_0 that minimizes $V_\theta + \theta d$. These particular choices of parameters $\bar{N}$ and h were found to offer a reasonable trade-off between computational effort and accuracy. It was numerically found that a reasonable estimate of the minimum of $V_\theta + \theta d$ is $\theta_0 = 1$.

Let our designed robust suboptimal control u^* be the optimal control u_1 for the ($\theta = 1$) risk-sensitive control problem. Then Theorem 1 states that the designed u^* control provides a guaranteed cost control solution so that

$$\sup_{Q \in \Xi_d} J(u^*, Q) \leq V_1 + d \approx 0.265$$

where $V_1 + d = 0.265$ was numerically calculated in this problem. To correctly interpret this performance bound, it should be noted that our approximation for V_1 is only as accurate as provided by our choice of $\bar{N}$ and h. Refining $\bar{N}$ and h will refine our performance bound.

Figure 1 shows the second component while in mode 1 of the resulting approximation of the resulting state-feedback robust controller u^*.

Although the presented robust suboptimal control design approach requires a computationally expensive line search over θ, this compares favorably with a more standard robust control design approach that would require a more complicated search over the admissible perturbation class Ξ_d.

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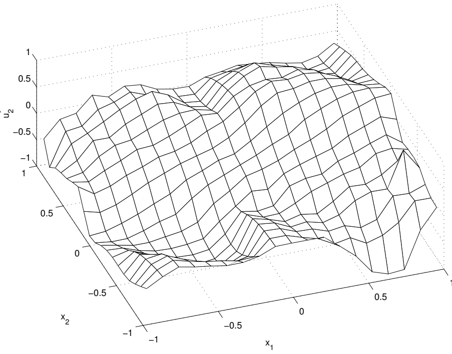

Fig. 1. Second component of u^* in mode 1, i.e., $r_t = 1$

REFERENCES

[1] I. R. Petersen, M. R. James, and P. Dupuis, "Minimax optimal control of stochastic uncertain systems with relative entropy constraints," *IEEE Transactions on Automatic Control*, vol. 45, pp. 398–412, 2000.

[2] V. A. Ugrinovskii and I. R. Petersen, "Finite horizon minimax optimal control of stochastic partially observed time varying uncertain systems," *Mathematics of Control, Signals and Systems*, vol. 12, no. 1, pp. 1–23, 1999.

[3] ——, "Minimax LQG control of stochastic partially observed uncertain systems," *SIAM J. Control Optim.*, vol. 40, no. 4, pp. 1189–1226, 2001.

[4] C. D. Charalambous and F. Rezaei, "Optimization of stochastic uncertain systems: large deviations and robustness," in *Proc. 42nd IEEE Conf. Decision Contr.*, vol. 4, 2003, pp. 4249–4253.

[5] Z. Pan and T. Başar, "H^∞ control of Markovian jump systems and solutions to associated piecewise-deterministic differential games," in *New Trends in Dynamic Games and Applications*, G. J. Olsder, Ed. Boston, MA: Birkhauser, 1995, pp. 61–92.

[6] V. A. Ugrinovskii, "Randomized algorithms for robust stability and guaranteed cost control of stochastic jump parameter systems with uncertain switching policies," *J. Optimization Theory Appl.*, vol. 124, no. 1, pp. 227–245, 2005.

[7] J. Xiong, J. Lam, H. Gao, and D. W. C. Ho, "On robust stabilization of Markovian jump systems with uncertain switching probabilities," *Automatica*, vol. 41, pp. 897–903, 2005.

[8] P. Dupuis and R. Ellis, *A Weak Convergence Approach to the Theory of Large Deviations*. Wiley, 1997.

[9] P. D. Pra, L. Meneghini, and W. J. Runggaldier, "Connections between stochastic control and dynamic games," *Mathematics of Control, Systems and Signals*, no. 2, pp. 303–326, 1996.

[10] I. R. Petersen, V. Ugrinovskii, and A. V. Savkin, *Robust Control Design using H^∞ Methods*. Springer-Verlag, 2000.

[11] T. Runolfsson, "Risk-sensitive control of stochastic hybrid systems on infinite time horizon," *Mathematical Problems in Engineering*, vol. 5, pp. 459–478, 2000.

[12] H. Kushner and P. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time*, 2nd ed. New York: Springer, 2001.

[13] P. Billingsley, *Convergence of Probability Measures*. John Wiley, 1968.

[14] E. Behrends, *Introduction to Markov Chains*. Braunschweig/Weisbaden: Vieweg, 2000.

[15] J. L. Doob, *Stochastic processes*. NY: John Wiley, 1953.

[16] R. S. Liptser and A. N. Shiryayev, *Statistics of Random Processes. II. Applications*. Springer-Verlag, 1978.

[17] M. K. Ghosh, A. Arapostathis, and S. I. Marcus, "Ergodic control of switching diffusions," *SIAM J. Control Optim.*, vol. 35, no. 6, pp. 1952–1988, 1997.

[18] T. Bielecki, D. Hernández-Hernández, and S. R. Pliska, "Value iteration for controlled Markov chains with risk sensitive cost criterion," in *Proceedings of the 1999 Conference on Decision and Control*, 1999, pp. 126–130.

[19] V. A. Ugrinovskii and H. R. Pota, "Decentralized control of power systems via robust control of uncertain Markov jump parameter systems," *Int. J. Contr.*, vol. 78, pp. 662–677, 2005.